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A Relativistic Framework to Estimate Clock Rates on the Moon

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Abstract

As humanity aspires to explore the solar system and investigate distant worlds such as the Moon, Mars, and beyond, there is a growing need to estimate and model the rate of clocks on these celestial bodies and compare them with the rate of standard clocks on Earth. According to Einstein's theory of relativity, the rate of a standard clock is influenced by the gravitational potential at its location and its relative motion. A convenient choice of local reference frames allows for the comparison of local time variations of clocks due to gravitational and kinematic effects. We estimate the rate of clocks on the Moon using a locally freely falling reference frame coincident with the center of mass of the Earth–Moon system. A clock near the Moon's selenoid ticks faster than one near the Earth's geoid, accumulating an extra 56.02 μs day^{-1} over the duration of a lunar orbit. This formalism is then used to compute the clock rates at Earth–Moon Lagrange points. Accurate estimation of the rate differences of coordinate times across celestial bodies and their intercomparisons using clocks on board orbiters at Lagrange points as time transfer links is crucial for establishing reliable communications infrastructure. This understanding also underpins precise navigation in cislunar space and on celestial bodies' surfaces, thus playing a pivotal role in ensuring the interoperability of various position, navigation, and timing systems spanning from Earth to the Moon and to the farthest regions of the inner solar system.

Unified Astronomy Thesaurus concepts: Celestial mechanics (211); Gravitation (661); Earth (planet) (439); The Moon (1692)

1. Introduction

More than 50 yr after the first lunar landing, a multinational consortium, which includes NASA, is working toward a return to the Moon under the Artemis Accords (Artemis Plan 2020; ESA 2022; NASA 2023). Our ability to explore distant worlds will require the design and development of a communication and navigation infrastructure within and beyond cislunar space. With the expectation of a significant increase in assets on the lunar surface and in cislunar space in the near future, developing a robust architecture for accurate position, navigation, and timing applications has become a matter of paramount interest.

Communication and navigation systems rely on a network of clocks that are synchronized to each other within a few tens of nanoseconds. As the number of assets on the lunar surface grows, synchronizing local clocks with higher precision using remote clocks on Earth becomes challenging and inefficient. An optimal solution would be to draw from the heritage of global navigation satellite systems by envisioning a system or constellation time common to all assets and then relating this time to clocks on Earth.

The relativistic framework using a generalized Fermi frame presented here enables us to compare clock rates on the Moon and cislunar Lagrange points with respect to clocks on Earth by using a metric appropriate for a locally freely falling frame such as the center of mass of the Earth–Moon system in the Sun's gravitational field (Fermi & Lincei 1922; Ashby & Bertotti 1986). The International Astronomical Union (IAU) resolutions provide a fully relativistic framework for transformations of coordinates (including time) and gravitational potentials and parameterizing potential coefficients using post-Newtonian potentials for constructing local reference systems for all celestial bodies in the solar system (IAU 2000a, 2000b; Soffel et al. 2003; Kaplan 2006). The relativistic celestial mechanics of the Earth–Moon system can also be described by adopting Jacobi coordinates within the framework of IAU resolutions (Kopeikin & Xie 2010). More recently, the IAU Resolutions Committee has approved proposals to establish a Standard Lunar Celestial System and Lunar Coordinate Time (IAU 2024). Here, we explicitly describe a framework to apply Einstein's theory of relativity for estimating and comparing clock rates to within an accuracy of a few nanoseconds a day on celestial bodies, constituting a restricted three-body problem.

The time measured by a clock at any given location is the proper time. Relativity of simultaneity implies that no two observers will agree on a given sequence of events if they are in different reference frames (Einstein 1996). In other words, clocks in different reference frames tick at different rates. The gravitational and motional effects affect the ticking rate of clocks when compared with "ideal" clocks that are at rest and sufficiently far away from any gravitating mass. For example, clocks farther away from Earth tick faster, and clocks in uniform motion will tick slower with respect to "ideal" clocks, and vice versa. Therefore, choosing an appropriate reference frame becomes essential for obtaining self-consistent results when comparing clocks on two celestial bodies. The gravitational effects on clocks and clock comparisons add another layer of complexity to synchronization challenges in deep-space communications (Burleigh et al. 2003; Burt et al. 2021).

In this paper, we mainly seek answers to the following questions: what is a good choice for the coordinate system that can be used to relate the proper times on the Earth and the Moon? What is an appropriate choice for the locations of ideal clocks on the surfaces of the Earth and Moon that makes it easier to compare their proper times? What is the proper time difference between clocks on the Moon and the Earth? What are the proper time differences between clocks located at the Earth–Moon Lagrange points and the Earth? The stability offered by Lagrange points provides a low acceleration noise environment for spacecraft with clocks. The relativistic corrections for such clocks can be precisely estimated, as their positions and velocities are well determined and can be used to compare the proper times of clocks on Earth and the Moon and in cislunar orbits.

In Section 2, we use the global positioning system (GPS) as an example to illustrate the relativistic effects on clocks if the Moon is treated just like an artificial satellite of the Earth and obtain a rough estimate for the clock rates on the Moon with respect to clocks on the geoid. Section 3 introduces a freely falling coordinate system with its center coinciding with the center of mass of the Earth and Moon. Section 4 compares the rate offset of a clock on the lunar surface to clocks on the geoid using this freely falling coordinate system, assuming the Moon is in a Keplerian orbit around the Earth. The results are compared with precise orbits for the Moon obtained using the latest planetary ephemerides, DE440 (Park et al. 2021). The time rate offsets at Earth–Moon Lagrange points L_1, L_2, L_3, and L_4/L_5 are also discussed in Section 5. Conclusions and future outlook are presented in Section 5. Appendices A and B introduce the framework for developing the metric used in all calculations. Appendix C justifies our assumptions of using a Keplerian model ignoring tidal effects, and a discussion in Appendix D establishes general covariance, meaning that the results are coordinate-independent.

2. Clocks in Orbit

An instructive example of establishing a coordinate time on Earth is the GPS time. The constellation clocks are set to beat at the average coordinate rate corresponding to clocks at rest on the surface of the rotating Earth by applying a “factory frequency offset” to the clocks before launch, which is (Ashby 2003)

$$\frac{\Delta f}{f} = \frac{3GM_e}{2ac^2} + \frac{\Phi_0}{c^2}, \tag{1}$$

where a is the satellite semimajor axis, G is Newton’s gravitational constant, M_e is the mass of the Earth, c is the speed of light in vacuum, and Φ_0 is the effective gravitational potential in the rotating frame, which is the sum of the static gravitational potential of the Earth and a centripetal contribution (Ashby 2003). The IAU defines a “Terrestrial Time” (TT) by adopting a fixed value for Φ_0/c^2 (IAU 2000a; Kaplan 2006):

$$-L_G = \frac{\Phi_0}{c^2} = -6.969290134(0) \times 10^{-10}. \tag{2}$$

If we simply substitute into Equation (1) the length of the semimajor axis of the Moon’s orbit, 3.84748×10^8 m, the “offset” becomes -6.7964×10^{-10}. To convert to a rate difference in microseconds per day, multiply by $86{,}400 \times 10^6\ \mu\text{s day}^{-1}$, yielding $58.721\ \mu\text{s day}^{-1}$. This does not include the effect of the Moon’s gravitational potential. Also, this approach can be questioned because it does not make sense to treat the Moon’s potential, from the point of view of an Earth-based inertial frame, as an Earth satellite; the Moon’s potential should be treated as a tidal potential. Nevertheless, a standard clock on an Earth satellite at the distance of the Moon would beat faster than a standard clock at rest on Earth by $58.721\ \mu\text{s day}^{-1}$, not including any effect from the gravitational potential of the Moon. This is a combination of Earth’s gravitational potential and second-order Doppler shifts at the orbiting satellite.

In addition, there are well-understood periodic effects arising from orbit eccentricity, with an additional contribution to the rate on the satellite clock given by (Ashby 2003)

$$\Delta = -\frac{2GM_e}{c^2}\left(\frac{1}{a} - \frac{1}{r}\right). \tag{3}$$

For a satellite in Keplerian orbit, the periodic contribution to the rate is

$$\Delta = \frac{2GM_e e}{c^2 a(1 - e^2)}(\cos(f) + e), \tag{4}$$

where e is the eccentricity and f is the true anomaly. Numerical evaluation of Equation (4) yields a value of $1.2695 \times 10^{-12}(\cos(f) + e)$ or $0.1097(\cos(f) + e)\ \mu\text{sday}^{-1}$. The time average of the combination $\cos(f) + e$ is zero, so this term does not contribute to the average rate.

This model uses an eccentric Keplerian orbit in a local inertial frame centered on Earth’s center of mass. The center of mass of the Earth and Moon approximately follows a Keplerian orbit. However, for the Earth–Moon system, one cannot have a Keplerian orbit in a coordinate system centered on the Earth and a Keplerian orbit in a coordinate system centered on the Earth–Moon center of mass with the same orbit parameters. There are also relativistic effects arising from changes in time and length scales, Lorentz contraction, and changes in tidal effects.

In the following sections, we shall investigate a local inertial system with an origin at the Earth–Moon center of mass. This is a freely falling inertial frame only in the Sun’s gravitational field. The reason for using such a frame is that the Earth and Moon are treated more or less equivalently; tidal potentials are due only to the Sun. By addressing relativistic effects in this simple system, it may be expected that the main relativistic corrections can be better understood. We work in a plane containing the Earth and Moon that is inclined with respect to the ecliptic plane. Calculations are carried out only to order c^{-2}. Contributions from the tidal potentials of other solar system bodies are left out. The metric signature is $(-1, 1, 1, 1)$ with Greek indices running from 0 to 3.

2.1. Local Frame for the Earth

In establishing a coordinate time on and near the Earth, two relativistic effects are compensated by adjusting the rates of standard clocks. These are (a) the gravitational potential at the geoid and (b) the second-order Doppler shift due to the Earth’s rotation. The gravitational potential at the equator can be estimated from existing models of Earth’s gravitational potential. Viewed from an Earth-centered inertial frame, the second-order Doppler shift is

$$\left(\frac{\delta f}{f}\right)_{\text{Dop}} = -\frac{\omega_e^2 a_e^2}{2c^2}, \tag{5}$$

where ω_e is Earth’s angular rotation rate and a_e is the equatorial radius. Because the Earth’s geoid is nearly a surface of effective hydrostatic equilibrium, all atomic clocks on the geoid beat at equal rates, and this rate can be calculated on the Earth’s geoid at the equator. The effective potential Φ_0 in Equation (2) represents the fractional rate difference between an atomic clock at rest at infinity if the Earth were the only celestial body and an atomic clock fixed on the geoid of the rotating Earth.

A locally inertial, freely falling reference frame can be constructed at the center of mass of the Earth. Such a construction (see Appendices A and B) gives the following expression for the fundamental scalar invariant ds^2 to order c^{-2} (Ashby & Bertotti 1986):

$$\begin{aligned} -ds^2 &= G_{\mu\nu}dx^\mu dx^\nu \\ &= -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2}{c^2}(\Phi_m + \Phi_s)|_{\rm tidal}\right)(dx^0)^2 \\ &+ \left(1 - \frac{2\Phi_e}{c^2} - \frac{2}{c^2}(\Phi_m + \Phi_s)|_{\rm tidal}\right)(dx^2 + dy^2 + dz^2), \end{aligned} \quad (6)$$

where dx^0, dx, dy, and dz are the changes in coordinate time and coordinate displacements. We have explicitly included the gravitational potential of the Earth, Φ_e, and the tidal potentials of the Moon and Sun, $(\Phi_m + \Phi_s)|_{\rm tidal}$, but have left out the small contributions from tidal potentials of other solar system bodies. If the timescale is adjusted by

$$dx^0 \to dx^0\left(1 - \frac{\Phi_0}{c^2}\right), \quad (7)$$

then the scalar invariant becomes

$$\begin{aligned} -ds^2 &= -\left(1 + \frac{2(\Phi_e - \Phi_0)}{c^2} + \frac{2}{c^2}(\Phi_m + \Phi_s)|_{\rm tidal}\right)(dx^0)^2 \\ &+ \left(1 - \frac{2\Phi_e}{c^2} - \frac{2}{c^2}(\Phi_m + \Phi_s)|_{\rm tidal}\right)(dx^2 + dy^2 + dz^2). \end{aligned} \quad (8)$$

With this adjustment in scale, apart from tidal effects, which average to near zero, clocks at rest on the geoid beat at the rate of International Atomic Time, defined by atomic clocks at rest on the geoid. Coordinate time suitable for use in navigation and timekeeping near the Earth's surface is then obtained by synchronizing clocks in the local inertial frame (Ashby & Allan 1979). The proper time on a clock at rest on the geoid then, apart from tidal contributions, beats at the rate of coordinate time because the term Φ_0 cancels the potential and second-order Doppler shifts on the geoid.

2.2. Local Frame for the Moon

While the Moon appears fairly rigid, it is nearly spherical due to hydrostatic equilibrium. One can imagine a locally inertial, freely falling reference frame with its origin at the Moon's center of mass; see Appendix A. Near the Moon, the scalar invariant will be

$$\begin{aligned} -ds^2 &= -\left(1 + \frac{2\Phi_m}{c^2} + \frac{2}{c^2}(\Phi_e + \Phi_s)|_{\rm tidal}\right)(dx^0)^2 \\ &+ \left(1 - \frac{2\Phi_m}{c^2} - \frac{2}{c^2}(\Phi_e + \Phi_s)|_{\rm tidal}\right)(dx^2 + dy^2 + dz^2). \end{aligned} \quad (9)$$

Omitting tidal terms for the moment, a standard clock at rest at the Moon's equator will be subject to the gravitational potential of the Moon and to time dilation from the Moon's rotation. Using a model of the Moon's potential (Bertone et al. 2021) that includes spherical harmonics of degree and order up to 350, the gravitational potential on the Moon's equator and second-order Doppler shift, respectively, are found to be approximately $\Phi_m|_{\theta=\pi/2} = -2.82101(7) \times 10^6$ m^2 s^{-2} and $-\omega_m^2 a_m^2/2 = -10.70118(14)$ m^{2}s^{-2}, where $a_m = 1{,}738{,}140(123)$ m is the equatorial radius and $\omega_m = 2.661621 \times 10^{-6}$ s^{-1} is the sidereal rotation rate of the Moon. The Moon's rotation is tidally locked to the Earth. Thus, we could define a constant L_m and a corresponding equipotential surface or "selenoid" for the Moon:

$$\begin{aligned} L_m &= -\frac{\Phi_{0m}}{c^2} = -\left(\frac{\Phi_m|_{\theta=\pi/2}}{c^2} - \frac{\omega_m^2 a_m^2}{2c^2}\right) \\ &= 3.13881(15) \times 10^{-11}. \end{aligned} \quad (10)$$

We may then, in analogy to the timescale change for the Earth, define a new timescale for the Moon such that the scalar invariant near the Moon becomes

$$\begin{aligned} -ds^2 &= -\left(1 + \frac{2(\Phi_m - \Phi_{0m})}{c^2} + \frac{2}{c^2}(\Phi_e + \Phi_s)|_{\rm tidal}\right)(dx^0)^2 \\ &+ \left(1 - \frac{2\Phi_m}{c^2} - \frac{2}{c^2}(\Phi_e + \Phi_s)|_{\rm tidal}\right)(dx^2 + dy^2 + dz^2). \end{aligned} \quad (11)$$

Then, apart from tidal effects, standard clocks at rest on an effective equipotential of the rotating Moon will beat at equal rates and can be used to define the rate of coordinate time on the Moon: $-ds^2 = -(dx^0)^2$.

3. Clock Rate Differences between Earth and the Moon

The Earth and the Moon orbit around their mutual center of mass in different Keplerian orbits. Meanwhile, the center of mass of the Earth–Moon system orbits around the Sun in an approximately Keplerian orbit. To calculate the rate differences between clocks on Earth and on the Moon, a fictitious locally freely falling inertial frame is introduced at the Earth–Moon center of mass. This makes it convenient to calculate the proper times elapsed on moving clocks in terms of Keplerian motions of the Earth and the Moon. The Sun's contribution is only tidal effects. If we omit the tidal potential of the Sun, the scalar invariant takes a simple form (Appendix B):

$$\begin{aligned} -ds^2 &= -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2\Phi_m}{c^2}\right)(dx^0)^2 \\ &+ \left(1 - \frac{2\Phi_e}{c^2} - \frac{2\Phi_m}{c^2}\right)(dx^2 + dy^2 + dz^2). \end{aligned} \quad (12)$$

Consider a clock fixed on the surface of the rotating geoid of Earth. Since the geoid is a surface of approximate hydrostatic equilibrium, if such clocks are viewed from the local inertial frame, they beat at the same rate, which can be evaluated at the equator. The proper time on the Earth-based clock becomes

$$\begin{aligned} -c^2 d\tau_e^2 &= -\left(1 + \frac{2\Phi_e}{c^2}|_{R_{\rm Eeq}} + \frac{2\Phi_m}{c^2}|_{R_{\rm Eeq}}\right)(dx^0)^2 \\ &+ \frac{(\boldsymbol{V}_e + \boldsymbol{v}_e)^2}{c^2}(dx^0)^2, \end{aligned} \quad (13)$$

where the equatorial radius of the Earth is denoted by $\boldsymbol{R}_{\rm Eeq}$, $\boldsymbol{V}_e$ is the velocity of the Earth's center of mass in the Earth–Moon coordinate system, and $\boldsymbol{v}_e$ represents the velocity of the clock on the equator due to Earth rotation. Expanding the velocity term, taking square roots of both sides and rearranging,

$$c d\tau_e = \left(1 + \frac{\Phi_e}{c^2}|_{R_{\rm Eeq}} - \frac{\mathbf{v}_e^2}{2c^2} + \frac{\Phi_m}{c^2}|_{R_{\rm Eeq}} - \frac{V_e^2}{2c^2} - \frac{\mathbf{V}_e \cdot \mathbf{v}_e}{c^2}\right)(dx^0). \tag{14}$$

The first two contributions can be identified with the quantity Φ_0. The contribution from the Moon's potential can be well approximated by setting

$$\frac{\Phi_m}{c^2}|_{R_{\rm Eeq}} = -\frac{GM_m}{c^2 D}, \tag{15}$$

where D is the Earth–Moon distance. The dot product term between velocities will depend on the specific position of the clock and will vary with a daily period; this variation is similar to the corrections to the gravitational potential contribution from the Moon arising from the fact that the clock is not at the center of the Earth. Omitting such contributions gives

$$c d\tau_e = \left(1 + \frac{\Phi_0}{c^2} - \frac{GM_m}{c^2 D} - \frac{V_e^2}{2c^2}\right)dx^0. \tag{16}$$

A similar argument applied to a clock fixed on the rotating Moon's surface of hydrostatic equilibrium gives the proper time,

$$c d\tau_m = \left(1 + \frac{\Phi_{0m}}{c^2} - \frac{GM_e}{c^2 D} - \frac{V_m^2}{2c^2}\right)dx^0, \tag{17}$$

where $\boldsymbol{V}_m$ is the velocity of the Moon's center of mass and Φ_{0m}, discussed above, is the combination of the Moon's gravitational potential on the selenoid and second-order Doppler shift due to the rotation of a clock on the Moon's equator. Therefore, the fractional frequency shift of a clock on the Moon's equator relative to a clock on Earth's equator is

$$\frac{d\tau_m - d\tau_e}{d\tau_e} = \frac{(GM_m - GM_e)}{c^2 D} + \frac{\Phi_{0m} - \Phi_0}{c^2} - \frac{1}{2c^2}(V_m^2 - V_e^2), \tag{18}$$

where GM_e and GM_m are the standard gravitational parameters for the Earth and Moon. The distance to the Moon from the Earth, for a Keplerian orbit, is given by (see Appendix C)

$$D = \frac{a(1 - e^2)}{1 + e\cos(f)}, \tag{19}$$

where f is the true anomaly plus possibly a constant, a is the length of the semimajor axis, $M_T = M_e + M_m$, and e is the eccentricity of the Moon's orbit. The following combination of quantities occurs frequently and can be reduced to a simpler expression,

$$D^2\dot{f}^2 + (\dot{D})^2 = \frac{GM_T}{a}\left(\frac{1 + 2e\cos(f) + e^2}{1 - e^2}\right), \tag{20}$$

where $\dot{D}$ and $\dot{f}$ are the time derivatives of D and f. The velocities have radial as well as transverse components. Consider first the quantity $\boldsymbol{V}_e^2$. The radial and transverse components of this velocity are

Table 1
Position-dependent Terms that Are Omitted in Calculating Rate Offsets

Position-dependent Term	Rate (μs day^{-1})	Period (days)
$GM_m a_e c^{-2} D^{-2}$	0.0002	1
$GM_e a_m c^{-2} D^{-2}$	0.0045	∼27
$\boldsymbol{V}_e \cdot \boldsymbol{v}_e c^{-2} \approx V_e \omega_e a_e c^{-2}$	0.0055	1
$\boldsymbol{V}_m \cdot \boldsymbol{v}_m c^{-2} \approx V_m \omega_m a_m c^{-2}$	0.0045	∼27

$$V_r = -\frac{M_m}{M_T}\dot{D}, \quad V_t = \frac{M_m}{M_T}D\dot{f}. \tag{21}$$

Therefore,

$$\frac{V_e^2}{2c^2} = \frac{1}{c^2}\left(\frac{M_m^2}{M_T^2}\right)(\dot{D}^2 + D^2\dot{f}^2) = \mu^2\frac{GM_T}{2ac^2}\left(\frac{1 + 2e\cos(f) + e^2}{1 - e^2}\right), \tag{22}$$

where $\mu = M_m/M_T = 0.012150$. A similar calculation for V_m yields

$$\frac{V_m^2}{2c^2} = (1 - \mu)^2\frac{GM_T}{2ac^2}\left(\frac{1 + 2e\cos(f) + e^2}{1 - e^2}\right). \tag{23}$$

The difference of the squares of the velocities is then

$$\frac{V_e^2 - V_m^2}{2c^2} = (2\mu - 1)\left(\frac{GM_T}{2ac^2}\right)\left(\frac{1 + 2e\cos(f) + e^2}{1 - e^2}\right), \tag{24}$$

where f is the true anomaly plus possibly a constant, a is the length of the semimajor axis, $M_T = M_e + M_m$, $\mu = M_m/M_T = 0.012150$, e is the eccentricity of the Moon's orbit, and GM_e and GM_m are the gravitational parameters of the Earth and the Moon, respectively. Using Equation (24) in Equation (18), we obtain

$$\frac{d\tau_m - d\tau_e}{d\tau_e} = \frac{(GM_m - GM_e)}{c^2 D} + \frac{\Phi_{0m} - \Phi_0}{c^2} - (1 - 2\mu)\left(\frac{GM_T}{2ac^2}\right)\left(\frac{1 + 2e\cos(f) + e^2}{1 - e^2}\right). \tag{25}$$

Now we shall discuss the small position-dependent terms that have been omitted. The actual distance from the center of the Moon to a clock on Earth's geoid is $|-\boldsymbol{D} + \boldsymbol{r}_e|$, where $\boldsymbol{r}_e$ is the vector from the Earth's center to the clock on the equator and $\boldsymbol{D}$ represents the vector from the center of the Earth to the center of the Moon. Then

$$-\frac{GM_m}{c^2|-\boldsymbol{D} + \boldsymbol{r}_e|} \approx -\frac{GM_m}{c^2 D} - \frac{GM_m}{c^2 D}\frac{\boldsymbol{D} \cdot \mathbf{r_e}}{D^2}. \tag{26}$$

Some of the position-dependent terms that we have not accounted for are given in Table 1. Since the Moon is tidally locked to the Earth, its center-of-mass velocity and the velocity of a clock on the selenoid will be highly correlated. Therefore, this term might give rise to a constant long-term average. Omitting the position-dependent terms, we have used the constants listed in Table 2 to evaluate the constant contribution

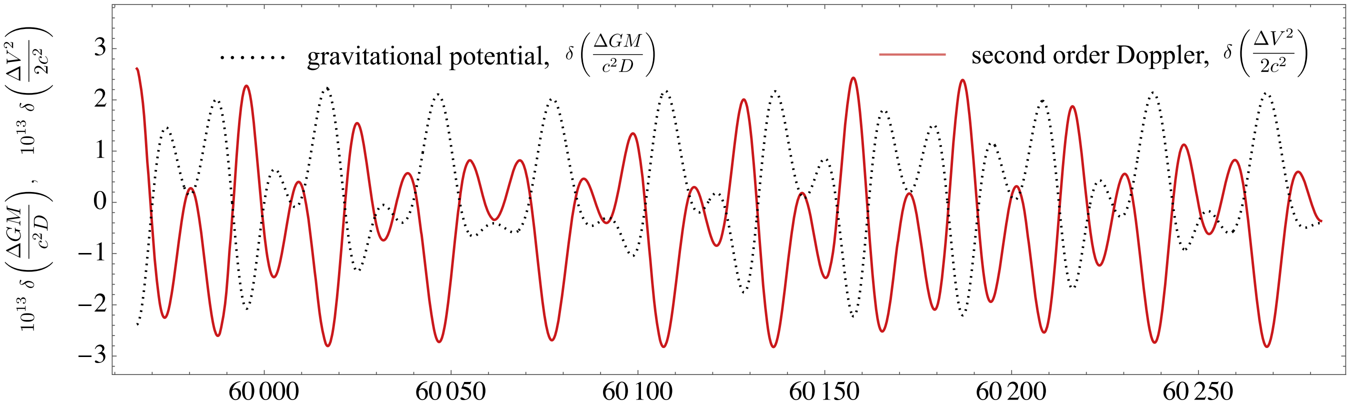

Figure 1. The residual of the gravitational potential and second-order Doppler for MJD 59965 (2023 January 21) to MJD 60282 (2023 December 4). $\Delta GM = GM_m - GM_e$ and $\Delta V^2 = V_m^2 - V_e^2$, where the subscripts refer to the Moon and the Earth, respectively, and the notation δ denotes that the quantities are residuals with the Keplerian prediction subtracted from values computed using DE440. The deviation from an ideal Keplerian orbit due to tidal dynamics causes the Moon to move faster as the separation between the Earth and the Moon decreases. The uncertainty on the estimated rate of the proper time on the Moon compared to the proper time of the Earth's geoid depends on how the above excursions are modeled and accounted for. Without clock corrections, the inaccuracy in time and position estimates can be as high as 10% compared to clocks in well-determined orbits.

Table 2
Constants and Values

Constant Parameter	Value
GM_e, Earth's gravitational parameter	$3.986004418(8) \times 10^{14}$ m^3 s^{-2} (Ries et al. 1992)
GM_m, Moon's gravitational parameter	$4.90280031(44) \times 10^{12}$ m^3 s^{-2} (Konopliv et al. 2013)
c, speed of light in vacuum	299,792,458(0) ms^{-1} (CODATA 2006; Mohr et al. 2012)
$-\Phi_0/c^2$, L_G	$6.969290134(0) \times 10^{-10}$, ∼60.2 μs day^{-1} (IAU 2000a)
$-\Phi_{0m}/c^2$, L_m	$3.13881(15) \times 10^{-11}$, ∼2.71 μs day^{-1}; see Equation (10)
e, eccentricity of the Moon's orbit	0.05490 (Daher et al. 2021)
a, Earth–Moon semimajor axis distance	3.84399×10^8 m (Daher et al. 2021)

and the amplitude of the periodic term. We find

$$\frac{d\tau_m - d\tau_e}{d\tau_e} = 6.48378(15) \times 10^{-10} - 1.25502518 \times 10^{-12}\cos(f). \quad (27)$$

Multiplying by $10^6 \times 86{,}400\ \mu$s to obtain a time difference per day gives

$$56.0199(12) - 0.10843417\cos(f)\ \mu\text{sday}^{-1}. \quad (28)$$

None of the above estimates include tidal effects. This omission is because as a tidal force pushes back and forth on a satellite, two other side effects have to be accounted for. These are a change in the satellite's position, which entails a change in the gravitational potential of the body about which the satellite is orbiting, and a change in the velocity of the satellite clock, which changes its second-order Doppler shift. The residuals of the gravitational potential and second-order Doppler shift for the Earth–Moon system obtained by subtracting the Keplerian model from that obtained from DE440 are graphed in Figure 1. Previous work on such problems has shown that these changes are of similar orders of magnitude. Summarizing, there are three contributions to the frequency shift of a clock in a satellite that are of similar orders of magnitude: (1) the perturbing tidal potential itself, (2) the perturbed position that changes the contribution from the main potential, and (3) the perturbed velocity that changes the time dilation contribution. Although the perturbing tidal potential can easily be estimated, calculating the other two contributions is more complicated. When the Keplerian model is compared with DE440 ephemerides, the effects of solar tides are plotted in Figure 2.

TT is realized by comparing rates of standard clocks at rest on Earth's geoid. Geocentric Coordinate Time (TCG) is defined by the rate of a standard clock falling along with the center of the Earth, not subject to Earth's potential (IAU 2000a). Then $d(\text{TT})/d(\text{TCG}) = 1 - L_G$ (Petit & Wolf 2005). By analogy, "Time on the Moon" (TM) could be defined by the rates of standard clocks at rest on the Moon's selenoid, and a "Moon-centric Coordinate Time (TCM) could be defined by the rate of a standard clock at the center of the Moon, not subject to the Moon's potential. Then $\boldsymbol{d}(\mathbf{TM})/\boldsymbol{d}(\mathbf{TCM}) = \mathbf{1} - \boldsymbol{L_m}$. If the periodic term in Equation (18) is denoted by

$$L_{Gm} = \frac{(GM_m - GM_e)}{c^2 D} - \frac{1}{2c^2}(V_m^2 - V_e^2), \quad (29)$$

the quantity L_{Gm} can be interpreted in terms of the approximate ratio between the rates of coordinate times for translating Equation (18) into the above terminology,

$$\frac{(1 + L_G)d(\text{TCM})}{(1 + L_m)d(\text{TCG})} = 1 + L_{Gm} + L_G - L_m. \quad (30)$$

Therefore, to $\mathcal{O}(c^{-2})$, $d(TCM)/d(TCG) = 1 + L_{Gm}$. When comparing clocks at a level of 10^{-18} or better in fractional frequency,

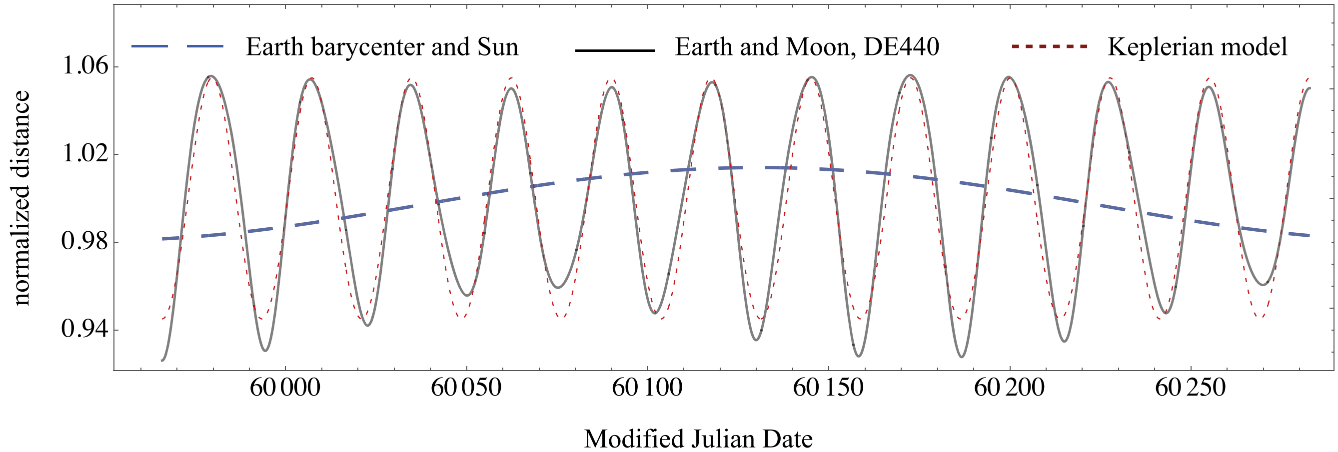

Figure 2. Earth–Moon distance compared with the Keplerian model. The normalized distance between the Earth and the Moon computed using the latest planetary ephemerides, DE440, is compared with the Keplerian model in the freely falling reference frame centered on the Earth–Moon barycenter. The tidal pull fluctuates at the perigee crossing for the Moon's orbit around the Earth as the Earth–Moon barycenter orbits the Sun. As a result, the actual Earth–Moon distance fluctuates compared to the Keplerian model at the perigee crossing. The tidal acceleration on the moon due to the Earth and Sun is given in Equation (A11). DE440 accounts for the first term in Equation (A11), whereas the Keplerian model does not include any tidal terms. The second term in Equation (A11) is much smaller than the first term and is due to the Sun's effect on the Earth–Moon barycenter. The phase offset between the Earth's orbit around the Sun and the amplitude modulation of the Earth–Moon distance is due to the inclination of the lunar orbit (∼5°) with respect to the equatorial coordinate system, with the *xy*-plane coinciding with the Earth's equator.

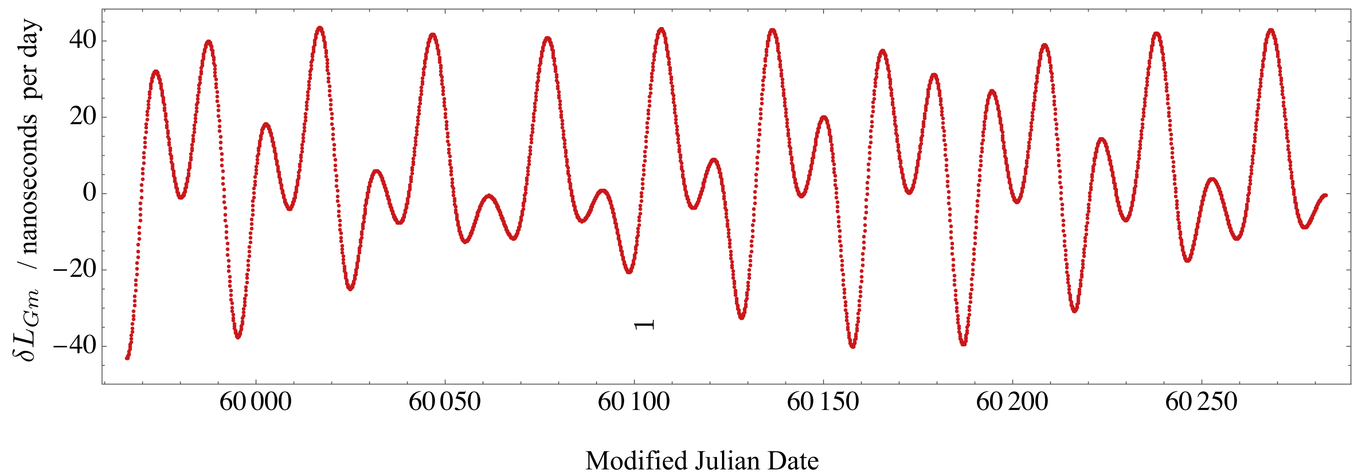

Figure 3. Time-varying component of the difference between TCG and TCM. δL_{Gm} is obtained by computing the residuals for Equation (30) using DE440 and the Keplerian model. The accumulated error in rate estimation throughout a lunar orbit can be as high as ∼75 ns. When the fluctuating components are modeled and accounted for, the clocks on the Moon can be synchronized to within a few nanoseconds a day with daily steers of the order of 1 ns. For comparison, the timescale at the National Institute of Standards and Technology (NIST) is routinely steered, and it can be as high as 250 ps day^{-1}. As a result, the Coordinated Universal Time (UTC) realized locally at Boulder, CO, UTC(NIST), stays within ±2 ns with respect to UTC over 1 yr.

this formula is an approximation. It may need to be reevaluated by including higher-order terms in c^{-2}, relativistic precessions, and terms arising due to the interaction of multipole moments of Earth and the Moon with tidal potentials of neighboring bodies. L_{Gm} in Equation (30) is a rate with a periodic contribution arising from Earth–Moon orbital eccentricity, which varies as the true anomaly $-1.49373 - 0.10967\cos(f)$ μs day^{-1}. The computed offset of L_{Gm} compared with data from DE440 is given in Figure 3.

It might appear that the second-order Doppler contribution to the rate difference depends on the coordinate system used. In the center-of-mass coordinates, a difference of squares of velocities appears, but in a system in which the Earth is at rest, the square of the relative velocity appears. Appendix D shows that contributions from the centrifugal potential, which occurs in a rotating coordinate system, resolve the apparent discrepancy.

4. Clocks at Earth–Moon Lagrange points

The Lagrange points or libration points offer a cost-effective and low-noise environment around which spacecraft can be in orbits that are slowly varying (Farquhar 1970). NASA and ESA have commissioned many observatories in the last 50 yr that have mostly been in Lissajous orbits around the Earth–Sun L_2 point (ISEE-3/ICE 1978; Wilkinson Microwave Anisotropy Probe (WMAP) 2001; Herschel Space Observatory 2009; Planck 2009; Gaia 2013; LISA Pathfinder 2015). NASA's James Webb Space Telescope is in a halo orbit with a fixed period (James Webb Space Telescope 2021). Halo orbits are more expensive than Lissajous because the spacecraft thrusters are to be fired more frequently to maintain orbits with fixed periods.

All five Lagrange points for the Earth–Sun system and the Earth–Moon are equilibrium points for objects with masses much smaller than the Earth, Sun, and Moon. L_1, L_2, and L_3 are saddle points that are unstable on a timescale of a few days as tiny departures from equilibrium grow with an e-folding time of a couple of days for the Earth–Moon system. The situation is very different for L_4 and L_5. The Coriolis forces acting on small masses at L_4 and L_5 of the Earth–Moon system allow a stability of the order of millions of years (Lissauer & Chambers 2008). We have not included the Coriolis effect in our treatment below as the resulting velocity components are small, so their contributions to the proper time are negligible.

For clocks in Lissajous or halo orbits, the relativistic corrections are due mainly to second-order Doppler terms from small velocity components, and the corresponding frequency offsets can be precisely determined using spacecraft attitude control data and ranging data from monitoring stations. Therefore, the rate of clocks at Lagrange points with respect to clocks on the Earth (and Moon) could play a vital role in synchronizing remote clocks in cislunar space. The Queqiao-1, 2 relay and radio astronomy satellites for the Chinese Lunar Exploration Program are currently in the halo and frozen orbits at Earth–Moon L_2 to support communications from the far side of the Moon (Wu et al. 2017). Whitley and Martinez discuss various satellite staging options in cislunar space (Whitley & Martinez 2016).

4.1. Clock at Lagrange Point L_1

We consider a clock at L_1 and compare its rate to a clock on Earth's surface. L_1 is between the Earth and the Moon and at a distance x_1D from the Moon. The net gravitational force toward the Moon supplies the radial acceleration of L_1, diminished by the centripetal acceleration due to the rotation of the Earth–Moon line to obtain $x_1 = 0.15093428(1)$ (Szebehely 1967). The metric, neglecting solar tides, is given by Equation (12). The motion is so slow that the potential terms in the last line can be neglected. The transverse and radial velocities of the clock due to the rotation of the Earth–Moon line are

$$V_{t(L1)}^2 = \left(\frac{df}{dt}\right)^2\left(\frac{M_e}{M_T} - x_1\right)^2 D^2;\; V_{r(L1)}^2 = \dot{D}^2\left(\frac{M_e}{M_T} - x_1\right)^2. \tag{31}$$

So, the total velocity squared is

$$V_{(L1)}^2 = \frac{GM_T(1-\mu-x_1)^2}{a(1-e^2)}(1 + 2e\cos(f) + e^2). \tag{32}$$

The proper time elapsed during a time interval dx^0 is

$$d\tau_{L1} = \frac{dx^0}{c}\left(1 - \frac{GM_e}{c^2D(1-x_1)} - \frac{GM_m}{c^2Dx_1} - \frac{GM_T(1-\mu-x_1)^2}{2c^2a(1-e^2)}(1 + 2e\cos(f) + e^2)\right). \tag{33}$$

For a clock on Earth's surface,

$$-ds^2 = -\left(1 - \frac{2GM_e}{c^2r_e}\left(1 - \frac{J_2a_e^2}{r_e^2}P_2(\cos\theta)\right) - \frac{2GM_m}{c^2D}\right) \times (cdx_0)^2 + \frac{V^2}{c^2}(cdT)^2, \tag{34}$$

where a_e is the equatorial radius of the Earth, J_2 is the constant describing the oblateness of the Earth, P_2 is the Legendre polynomial of degree 2, and θ is the latitude on the Earth's surface at a distance r_e from the center of the Earth. We approximated the distance from the Moon's center by D since the position of the comparison clock on Earth's surface is unspecified. The total velocity of the clock on Earth's surface is composed of orbital velocity plus Earth's rotational velocity:

$$\boldsymbol{V} = \boldsymbol{V}_{\rm orbit} + \omega_e \times \mathbf{r_e}, \tag{35}$$

$$V^2 = V_{\rm orbit}^2 + 2\boldsymbol{V}_{\rm orbit}\cdot(\omega_e\times\mathbf{r_e}) + (\omega_e\times\mathbf{r_e})^2, \tag{36}$$

where ω_e is the rate of Earth's rotation. We set aside the cross-term since the position of the Earth-based clock is changing rapidly, and this term averages down. The centripetal term is grouped with the Earth potential term. The square of the orbital velocity is composed of the squared radial velocity and the squared transverse velocity:

$$V_{\rm orbit}^2 = \dot{D}^2\left(\frac{M_m}{M_T}\right)^2 + \left(\frac{df}{dT}\right)^2\left(\frac{M_m}{M_T}\right)^2 D^2 = \mu^2\frac{GM_T}{a(1-e^2)}(1 + 2e\cos(f) + e^2). \tag{37}$$

The Earth's potential contribution plus the rotational term can be replaced by $2\Phi_0/c^2$. The proper time interval for the clock on Earth is then approximately

$$d\tau_e = \frac{dx^0}{c}\left(1 + \frac{\Phi_0}{c^2} - \frac{GM_m}{c^2D} - \mu^2\frac{GM_T}{2c^2a(1-e^2)} \times (1 + 2e\cos(f) + e^2)\right), \tag{38}$$

where $cdT = dX^0$. The fractional rate difference is then

$$\frac{d\tau_{L1} - d\tau_e}{d\tau_e} = -\frac{GM_e}{c^2D(1-x_1)} - \frac{GM_m}{c^2Dx_1} - \frac{\Phi_0}{c^2} - \frac{GM_T(1-\mu-x_1)^2}{2c^2a(1-e^2)}(1 + 2e\cos(f) + e^2) + \frac{GM_m}{c^2D} + \mu^2\frac{GM_T}{2c^2a(1-e^2)}(1 + 2e\cos(f) + e^2). \tag{39}$$

Evaluating this result for x_1 using values given in Table 2 gives

$$\frac{d\tau_{L1} - d\tau_e}{d\tau_e} = 6.7838449(12)\times 10^{-10} - 1.2426049 \times 10^{-12}\cos(f), \tag{40}$$

$$\frac{d\tau_{L1} - d\tau_e}{d\tau_e} = 58.612420(12) - 0.10736106\cos(f)\ \mu{\rm s\,day}^{-1}. \tag{41}$$

The result is dominated by the term in Φ_0 because the clock is high up in the Earth's potential.

4.2. Clock at Lagrange Point L_2

The Lagrange point is at a distance x_2D on the side of the Moon away from Earth. Gravitational forces due to both Earth and the Moon are toward the Earth and supply the force necessary for the centrifugal acceleration, diminished by the radial acceleration to obtain $x_2 = 0.16783274(1)$. The velocity squared of the clock, composed of radial and transverse velocities squared, is

$$V_{(L2)}^2 = \frac{GM_T(1-\mu+x_2)^2}{a(1-e^2)}(1+2e\cos(f)+e^2). \quad (42)$$

The proper time on a clock at L_2 is then

$$d\tau_{L2} = \frac{dx^0}{c}\left(1 - \frac{GM_e}{c^2D(1+x_2)} - \frac{GM_m}{c^2Dx_2} - \frac{GM_T(1-\mu+x_2)^2}{2c^2a(1-e^2)}(1+2e\cos(f)+e^2)\right). \quad (43)$$

For a comparison clock on Earth, the analysis is the same as for the clock at L_1. Therefore, the fractional rate difference is

$$\frac{d\tau_{L2}-d\tau_e}{d\tau_e} = -\frac{GM_e}{c^2D(1+x_2)} - \frac{GM_m}{c^2Dx_2} - \frac{\Phi_0}{c^2} - \frac{GM_T(1-\mu+x_2)^2}{2c^2a(1-e^2)}(1+2e\cos(f)+e^2) + \frac{GM_m}{c^2D} + \mu^2\frac{GM_T}{2c^2a(1-e^2)}(1+2e\cos(f)+e^2). \quad (44)$$

Evaluating this result numerically gives

$$\frac{d\tau_{L2}-d\tau_e}{d\tau_e} = 6.7846805(12)\times 10^{-10} - 1.4416552 \times 10^{-12}\cos(f), \quad (45)$$

$$\frac{d\tau_{L2}-d\tau_e}{d\tau_e} = 58.619639(12) - 0.12445590\cos(f)\ \mu\text{sday}^{-1}. \quad (46)$$

The clock beats faster since L_2 is farther out in Earth's gravitational potential.

4.3. Clock at Lagrange Point L$_3$

L_3 is behind the Earth, away from the Moon, and so of minor interest for cislunar considerations. We include it here for completeness. Let the distance of L_3 from the Earth be $D(1-x_3)$. The centripetal acceleration of this point is supplied by the sum of the gravitational forces due to the Earth and the Moon to give $x_3 = 0.0070879383(1)$. The proper time on a clock at L_3, including its transverse and longitudinal velocities, will be

$$d\tau_{L3} = \frac{dx^0}{c}\left(1 - \frac{GM_e}{c^2D(1-x_3)} - \frac{GM_m}{c^2D(2-x_3)} - \frac{GM_T}{2c^2a(1-e^2)}(1+2e\cos(f)+e^2)(1-x+\mu)^2\right). \quad (47)$$

The fractional difference is

$$\frac{d\tau_{L3}-d\tau_e}{d\tau_e} = -\frac{GM_e}{c^2D(1-x_3)} - \frac{GM_m}{c^2D(2-x_3)} - \frac{GM_m}{c^2D(2-x_3)} - \frac{GM_T}{2c^2a(1-e^2)}(1+2e\cos(f)+e^2)(1-x_3+\mu)^2 - \frac{\Phi_0}{c^2} + \frac{GM_m}{c^2D} + \frac{\mu^2GM_T}{2c^2}\frac{(1+2e\cos(f)+e^2)}{a(1-e^2)}. \quad (48)$$

Evaluation of this expression gives

$$\frac{d\tau_{L3}-d\tau_e}{d\tau_e} = 6.7942694(12)\times 10^{-10} - 1.28439230 \times 10^{-12}\cos(f), = 58.702488(12) - 0.11097149\cos(f)\ \mu\text{sday}^{-1}. \quad (49)$$

4.4. Clock at Lagrange Point L$_4$ or L$_5$

The clock is equidistant from Earth and the Moon. The total velocity squared is the sum of the radial velocity squared and the transverse velocity squared:

$$V_{(L4)}^2 = (\dot{D})^2 + \left(\frac{df}{dt}\right)^2 D^2 = \frac{GM_T}{a(1-e^2)}(1+2e\cos(f)+e^2). \quad (50)$$

The proper time on the clock is given by

$$-d\tau_{L4}^2 = -\left(1 - \frac{2GM_e}{c^2D} - \frac{2GM_m}{c^2D} - \frac{V_{(L4)}^2}{c^2}\right)dx_0^2. \quad (51)$$

This reduces to

$$d\tau_{L4} = \left(1 - \frac{GM_T}{c^2D} - \frac{GM_T}{2c^2a(1-e^2)}(1+2e\cos(f)+e^2)\right)dx_0. \quad (52)$$

Analysis of the comparison clock on Earth's surface is the same as for clocks at L_1 or L_2. The fractional rate difference is

$$\frac{d\tau_{L4}-d\tau_e}{d\tau_e} = -\frac{GM_T}{c^2D} - \frac{\Phi_0}{c^2} + \frac{GM_m}{c^2D} - \frac{GM_T}{2ac^2(1-e^2)} \times (1-\mu^2)(1+2e\cos(f)+e^2). \quad (53)$$

Evaluating this result numerically gives

$$\frac{d\tau_{L4}-d\tau_e}{d\tau_e} = 6.7948239(12)\times 10^{-10} - 1.27837388\times 10^{-12}\cos(f), \quad (54)$$

$$\frac{d\tau_{L4}-d\tau_e}{d\tau_e} = 58.707278(12) - 0.11045150\cos(f)\ \mu\text{sday}^{-1}. \quad (55)$$

5. Conclusions

We presented a model based on Keplerian orbits for estimating clock rates on the Moon and rates of clocks at Lagrange points in cislunar space. We used values for Keplerian orbit parameters that can be looked up; the only parameters that fit were the times of periapsis passage. The main numerical results obtained using our approach are given in Table 3. We assumed a fixed eccentricity and fixed value for the semimajor axis for the Moon's orbit around the Earth, as the present-day values for these parameters are very slowly varying (Daher et al. 2021).

The planetary ephemeris DE440 was used to calculate the potentials and velocities of Equation (25); the difference between the DE440 calculation and the Keplerian model calculations is only of the order of a few nanoseconds per day.

Table 3
Clock Rates Computed for Various Points of Interest

Quantity	Location	Rate (μs day^{-1})
$(d\tau_m/d\tau_e) - 1$	Lunar surface	$56.0199(12) - 0.10843417\cos(f)$
$(d\tau_{L_1}/d\tau_e) - 1$	L_1	$58.612420(12) - 0.10736106\cos(f)$
$(d\tau_{L_2}/d\tau_e) - 1$	L_2	$58.619639(12) - 0.12445590\cos(f)$
$(d\tau_{L_3}/d\tau_e) - 1$	L_3	$58.702488(12) - 1.28439230\cos(f)$
$(d\tau_{L_4}/d\tau_e) - 1$	L_4/L_5	$58.707278(12) - 0.11045150\cos(f)$

Such differences are due to tidal potentials arising from solar system bodies. Tidal effects can be readily modeled using available orbit data and added as corrections to the Keplerian model for synchronizing remote clocks on the Moon to within a few hundred ps or better. Changes in time coordinates entail changes in length scale, which should be of higher order than the c^{-2} effects we have considered here (see, for example, the length scale change in Equation (A4)). Our analysis should serve as a reference point to ensure the accuracy and consistency of results from numerical models when comparing clocks at the 10^{-18} level of fractional frequency. Numerical methods may be more suitable for estimating and comparing other effects that we have not considered in this work, such as relativistic precessions and post-Newtonian corrections. These effects may be significant at the 10^{-19} level in fractional frequency (less than a few tenths of a ps day^{-1}).

This approach is also useful in calculating time comparisons between Earth and clocks in the neighborhood of other solar system bodies such as Mars. The available spherical harmonic gravity potential for Mars allows an estimate of the quantity L_M for Mars that includes the average equatorial potential and rotational effects, analogous to L_G for Earth. In the case of Mars, the only available coordinate systems for the description of the problem are barycentric coordinates. The Earth–Mars rate difference is dominated by the difference in the Sun's gravitational potential at the two locations. Keplerian models, as well as computations using DE440, can be usefully compared; this will be the subject of a future paper. Spatial transformations accompanying time transformations also remain to be examined as part of future work.

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Appendix A
Fermi Coordinates with the Origin at the Center of the Moon

The Moon's center of mass is in freefall, and therefore its path is a geodesic. We disregard the effects of the Moon's multipole moments interacting with the tidal forces from the Earth and the Sun, as they are less than 10^{-19} in fractional frequency. It is useful to construct Fermi coordinates with the origin at this point, since then the only forces on an object in the neighborhood of the Moon due to external bodies are tidal forces. In this coordinate system, the Christoffel symbols due to external bodies are all zero at the origin, while contributions to the Christoffel symbols from the Moon itself must be "effaced," or discarded, since they are infinite and such terms cannot cause acceleration of the Moon itself. The following calculation is taken only to order c^{-2}. The geodesic in question is complicated because the Earth–Moon system orbits the Sun in an approximately Keplerian orbit, while the Moon and Earth revolve around each other in a different, approximately Keplerian orbit; this latter orbit is perturbed by the Sun's tidal potential and so is not known analytically. We can still construct Fermi coordinates since many unknown quantities cancel out. Here, we show how the metric given in Equation (12) arises.

The combined gravitational potential of the Sun, Earth, and Moon is given by

$$\Phi = \Phi_e + \Phi_s + \Phi_m, \tag{A1}$$

where the subscripts e, s, and m represent the potentials of the Earth, Sun, and Moon, respectively. Beginning with the metric in the solar system barycentric coordinates,

$$-ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)(dX^0)^2 + \left(1 - \frac{2\Phi}{c^2}\right)(dX^2 + dY^2 + dZ^2), \tag{A2}$$

where dX^0 is the time and dX, dY, and dZ are the space coordinate displacements in the barycentric coordinate system. Lowercase letters will be reserved for corresponding quantities in the local Fermi normal coordinate system.

We give the transformation equations between barycentric coordinates and Fermi normal coordinates with the center at the Moon as follows (Ashby & Bertotti 1986):

$$X^0 = \int_0^{x^0}\left(1 - \frac{\Phi_e(m) + \Phi_s(m)}{c^2} + \frac{V(m)^2}{2c^2}\right)dx^0 + \frac{\boldsymbol{V}(m)\cdot\boldsymbol{r}}{c}; \tag{A3}$$

$$X^k = X^k(m) + x^k\left(1 + \frac{\Phi_e(m) + \Phi_s(m)}{c^2} - \frac{\boldsymbol{A}(m)\cdot\boldsymbol{r}}{c^2}\right) + \frac{r^2 A(m)^k}{2c^2} + \frac{V(m)^k \boldsymbol{V}(m)\cdot\boldsymbol{r}}{2c^2}. \tag{A4}$$

Here, the notation (m) as in $\boldsymbol{V}(m)$ represents quantities evaluated at the Moon's center of mass. The quantity $V(m)$ is the magnitude of the Moon's velocity. Transformation coefficients can be derived and are

$$\frac{\partial X^0}{\partial x^0} = 1 - \frac{\Phi_s(m) + \Phi_e(m)}{c^2} + \frac{V(m)^2}{2c^2} + \frac{\boldsymbol{A}(m)\cdot\boldsymbol{r}}{c^2}, \quad \frac{\partial X^0}{\partial x^k} = \frac{V(m)^k}{c}, \ \frac{\partial X^k}{\partial x^0} = \frac{V(m)^k}{c}, \tag{A5}$$

$$\frac{\partial X^k}{\partial x^j} = \delta^k_j\left(1 + \frac{\Phi_s(m) + \Phi_e(m)}{c^2} - \frac{\boldsymbol{A}(m)\cdot \boldsymbol{r}}{c^2}\right) + \frac{V(m)^k V(m)^j}{2c^2} - \frac{x^k A(m)^j - x^j A(m)^k}{c^2}. \tag{A6}$$

Transformation of the metric tensor is accomplished with the usual formula,

$$g_{\alpha\beta} = \frac{\partial X^\mu}{\partial x^\alpha}\frac{\partial X^\nu}{\partial x^\beta}G_{\mu\nu}, \tag{A7}$$

where the summation convention for repeated indices applies. Thus, for the time-time component of the metric tensor in the freely falling frame,

$$\begin{aligned} g_{00} &= \left(\frac{\partial X^0}{\partial x^0}\right)^2 G_{00} + \sum_{j=1}^{j=3}\left(\frac{\partial X^j}{\partial x^0}\right)^2 G_{jj} \\ &= \left(1 - \frac{\Phi_s(m) + \Phi_e(m)}{c^2} + \frac{V(m)^2}{2c^2} + \frac{\boldsymbol{A}(m)\cdot\boldsymbol{r}}{c^2}\right)^2 \\ &\times(-1)\left(1 + \frac{2(\Phi_e + \Phi_m + \Phi_s)}{c^2}\right) + \frac{V(m)^2}{c^2}\left(1 - \frac{2(\Phi_e + \Phi_m + \Phi_s)}{c^2}\right). \end{aligned} \tag{A8}$$

Expanding and keeping the terms of order c^{-2},

$$\begin{aligned} g_{00} &= -1 + \tfrac{2\Phi_m}{c^2} + \tfrac{2(\Phi_e + \Phi_s)}{c^2} \\ &- \tfrac{2(\Phi_e(m) + \Phi_s(m))}{c^2} + \tfrac{2\boldsymbol{A}(m)\cdot\boldsymbol{r}}{c^2}. \end{aligned} \tag{A9}$$

Except for the Moon's potential, the terms in the last line add up to the solar tidal potential; for expanding $\Phi_e + \Phi_s$ about the origin and using

$$A(m)^k = -\frac{\partial(\Phi_e + \Phi_s)}{\partial X^k}, \tag{A10}$$

we find

$$\begin{aligned} &\frac{\Phi_e}{c^2} + \frac{\Phi_s}{c^2} - \frac{\Phi_e(m) + \Phi_s(m)}{c^2} + \frac{\boldsymbol{A}_{cm}\cdot\boldsymbol{r}}{c^2} = \frac{\Phi_t}{c^2} \\ &= -\frac{GM_e}{c^2 r_e^5}\frac{3(\boldsymbol{r}_{\mathbf{e}}\cdot\boldsymbol{r})^2 - r_e^2 r^2}{2} - \frac{GM_s}{c^2R^5}\frac{3(\boldsymbol{R}\cdot\boldsymbol{r})^2 - R^2r^2}{2}, \end{aligned} \tag{A11}$$

where $\boldsymbol{R}$ is the vector from the Sun to the center of the Moon, $\boldsymbol{r}_{\mathbf{e}}$ is the vector from the Earth to the center of the Moon, and $\boldsymbol{r}$ is the vector from the center of the Moon to the observation point in local Fermi normal coordinates. Equation (A11) gives the total tidal potential Φ_t/c^2 in the vicinity of the Moon due to the Earth and the Sun. Thus,

$$g_{00} = -\left(1 + \frac{2\Phi_m}{c^2} + \frac{2\Phi_t}{c^2}\right). \tag{A12}$$

For the spatial component g_{11}, we have

$$\begin{aligned} g_{11} &= \left(\frac{\partial X^0}{\partial x^1}\right)^2 G_{00} + \sum_{j=1}^{j=3}\left(\frac{\partial X^j}{\partial x^1}\right)^2 G_{jj} \\ &= \left(\frac{V(m)^1}{c}\right)^2(-1) + \left(\frac{\partial X^1}{\partial x^1}\right)^2 G_{11} \\ &= 1 - \frac{2\Phi_m}{c^2} - \frac{2\Phi_t}{c^2}, \end{aligned} \tag{A13}$$

where we have again expanded and kept only terms of order c^{-2}. Similarly,

$$g_{22} = g_{33} = g_{11}. \tag{A14}$$

The metric component g_{12} is given by

$$g_{12} = \frac{\partial X^0}{\partial x^1}\frac{\partial X^0}{\partial x^2}G_{00} + \sum_j \frac{\partial X^j}{\partial x^1}\frac{\partial X^j}{\partial x^2}G_{jj}. \tag{A15}$$

Keeping only terms of order c^{-2}, this becomes

$$\begin{aligned} g_{12} &= -\frac{V(m)^1V(m)^2}{c^2} \\ &+ \left(\frac{V(m)^1V(m)^2}{2c^2} + \frac{V(m)^2V(m)^1}{2c^2}\right) = 0. \end{aligned} \tag{A16}$$

Similarly,

$$g_{13} = g_{23} = 0. \tag{A17}$$

Summarizing, the scalar invariant with the origin at the Moon's center is

$$\begin{aligned} -ds^2 &= -\left(1 + \frac{2\Phi_m}{c^2} + \frac{2\Phi_t}{c^2}\right)(dx^0)^2 \\ &+ \left(1 - \frac{2\Phi_m}{c^2} - \frac{2\Phi_t}{c^2}\right)(dx^2 + dy^2 + dz^2). \end{aligned} \tag{A18}$$

The speed of light c is a defined quantity, which does not change when transforming coordinates. However, because the timescale changes, the length scale will also change. A quantity such as Φ_m/c^2 has been carried forward from barycentric coordinates, and one might question whether it should change due to time and length scale changes. However, such quantities are already of order c^{-2}, and any such changes would be of higher order and are therefore negligible. In these coordinates, contributions to Christoffel symbols of the second kind due to external bodies are zero since tidal potentials have been neglected.

Appendix B
Construction of Freely Falling Center-of-mass Frame

We illustrate the method of construction of a freely falling, locally inertial frame by constructing such a frame at the center of mass of the Earth–Moon system, assuming this point revolves around the Sun in an elliptical Keplerian orbit. We keep contributions only to order c^{-2} and neglect tidal contributions from solar system bodies other than the Earth, Moon, and Sun. We also neglect precessions.

The metric in isotropic barycentric coordinates including only the Earth, Moon, and Sun is

$$\begin{aligned} -ds^2 &= -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2\Phi_m}{c^2} + \frac{2\Phi_s}{c^2}\right)(dX^0)^2 \\ &+ \left(1 - \frac{2\Phi_e}{c^2} - \frac{2\Phi_m}{c^2} - \frac{2\Phi_s}{c^2}\right)(dX^2 + dY^2 + dZ^2), \end{aligned} \tag{B1}$$

where the gravitational potentials of the Earth, Moon, and Sun are denoted by subscripts e, m, and s, respectively. We use uppercase letters to denote quantities in barycentric coordinates and lowercase letters for quantities in the freely falling center of the mass frame. We are interested in a test particle at the Earth–Moon center of mass. The local time coordinate x^0 is determined by the proper time on an ideal clock at the center of mass. Consider the transformation of coordinates (Ashby & Bertotti 1986)

$$X^0 = \int_0^{x^0}\left(1 - \frac{\Phi_s(cm)}{c^2} + \frac{V(cm)^2}{2c^2}\right)dx^0 + \frac{\boldsymbol{V}_{cm}\cdot\boldsymbol{r}}{c}; \quad \text{(B2)}$$

$$X^k = \int_0^{x^0}\frac{V_{cm}^k}{c}dx^0 + x^k\left(1 + \frac{\Phi_s(cm)}{c^2} - \frac{\boldsymbol{A}_{cm}\cdot\boldsymbol{r}}{c^2}\right) + \frac{r^2A_{cm}^k}{2c^2} + \frac{V(cm)^k\boldsymbol{V}_{cm}\cdot\boldsymbol{r}}{2c^2}. \quad \text{(B3)}$$

Here $\boldsymbol{V}_{cm}$ and $\boldsymbol{A}_{cm}$ represent the velocity and acceleration of the center of mass.

The transformation coefficients are easily obtained from the above coordinate transformations and are

$$\frac{\partial X^0}{\partial x^0} = 1 - \frac{\Phi_s(cm)}{c^2} + \frac{V(cm)^2}{2c^2} + \frac{\boldsymbol{A}_{cm}\cdot\boldsymbol{r}}{c^2},$$
$$\frac{\partial X^0}{\partial x^k} = \frac{V(cm)^k}{c}, \quad \frac{\partial X^k}{\partial x^0} = \frac{V(cm)^k}{c}, \quad \text{(B4)}$$

$$\frac{\partial X^k}{\partial x^j} = \delta_j^k\left(1 + \frac{\Phi_s(cm)}{c^2} - \frac{\boldsymbol{A}_{cm}\cdot\boldsymbol{r}}{c^2}\right) + \frac{V(cm)^kV(cm)^j}{2c^2} - \frac{x^kA_{cm}^j - x^jA_{cm}^k}{c^2}. \quad \text{(B5)}$$

The metric component g_{00} in the center-of-mass frame, using Equation (A7), is

$$g_{00} = \left(\frac{\partial X^0}{\partial x^0}\right)^2 G_{00} + \sum_{j=1}^{j=3}\left(\frac{\partial X^j}{\partial x^0}\right)^2 G_{jj}$$
$$= \left(1 - \frac{\Phi_s(cm)}{c^2} + \frac{V(cm)^2}{2c^2} + \frac{\boldsymbol{A}_{cm}\cdot\boldsymbol{r}}{c^2}\right)^2 (-1)\left(1 + \frac{2(\Phi_e + \Phi_m + \Phi_s)}{c^2}\right) + \frac{V(cm)^2}{c^2}\left(1 - \frac{2(\Phi_e + \Phi_m + \Phi_s)}{c^2}\right). \quad \text{(B6)}$$

Expanding and keeping terms of order c^{-2},

$$g_{00} = -1 + \frac{2(\Phi_e + \Phi_m)}{c^2} + \frac{2\Phi_s}{c^2} - \frac{2\Phi_s(cm)}{c^2} + \frac{2\boldsymbol{A}_{cm}\cdot\boldsymbol{r}}{c^2}. \quad \text{(B7)}$$

The last three terms in the last line of Equation (B6) add up to twice the solar tidal potential; for expanding Φ_s about the center-of-mass point and using

$$A_{cm}^k = -\frac{\partial\Phi_s}{\partial X^k}, \quad \text{(B8)}$$

we find

$$\frac{\Phi_s}{c^2} - \frac{\Phi_s(cm)}{c^2} + \frac{\boldsymbol{A}_{cm}\cdot\boldsymbol{r}}{c^2} = -\frac{GM_s}{c^2R^5}\frac{3(\boldsymbol{R}\cdot\boldsymbol{r})^2 - R^2r^2}{2}, \quad \text{(B9)}$$

where $\boldsymbol{R}$ is the vector from the center of the Sun to the center-of-mass point. We denote the solar tidal potential by

$$\Phi_{st} = -\frac{GM_s}{R^5}\left(\frac{3(\boldsymbol{R}\cdot\boldsymbol{r})^2 - R^2r^2}{2}\right). \quad \text{(B10)}$$

Then

$$g_{00} = -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2\Phi_m}{c^2} + \frac{2\Phi_{st}}{c^2}\right). \quad \text{(B11)}$$

For the spatial component g_{11}, we have

$$g_{11} = \left(\frac{\partial X^0}{\partial x^1}\right)^2 G_{00} + \sum_{j=1}^{j=3}\left(\frac{\partial X^j}{\partial x^1}\right)^2 G_{jj}$$
$$= \left(\frac{V(cm)^1}{c}\right)^2(-1) + \left(\frac{\partial X^1}{\partial x^1}\right)^2 G_{11}$$
$$= 1 - \frac{2\Phi_e}{c^2} - \frac{2\Phi_m}{c^2} - \frac{2\Phi_{st}}{c^2}, \quad \text{(B12)}$$

where we have again expanded and kept only terms of order c^{-2}. Similarly,

$$g_{22} = g_{33} = 1 - \frac{2\Phi_e}{c^2} - \frac{2\Phi_m}{c^2} - \frac{2\Phi_{st}}{c^2}. \quad \text{(B13)}$$

The metric component g_{12} is given by

$$g_{12} = \frac{\partial X^0}{\partial x^1}\frac{\partial X^0}{\partial x^2}G_{00} + \sum_j\frac{\partial X^j}{\partial x^1}\frac{\partial X^j}{\partial x^2}G_{jj}. \quad \text{(B14)}$$

Keeping only terms of order c^{-2}, this becomes

$$g_{12} = -\frac{V(cm)^1V(cm)^2}{c^2} + \left(\frac{V(cm)^1V(cm)^2}{2c^2} + \frac{V(cm)^2V(cm)^1}{2c^2}\right) = 0. \quad \text{(B15)}$$

Similarly,

$$g_{13} = g_{23} = 0. \quad \text{(B16)}$$

Summarizing, the scalar invariant in the center-of-mass system is

$$-ds^2 = -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2\Phi_m}{c^2} + \frac{2\Phi_{st}}{c^2}\right)(dx^0)^2 + \left(1 - \frac{2\Phi_e}{c^2} - \frac{2\Phi_m}{c^2} - \frac{2\Phi_{st}}{c^2}\right)(dx^2 + dy^2 + dz^2). \quad \text{(B17)}$$

The speed of light c is a defined quantity, which does not change when transforming coordinates. However, because the timescale changes, the length scale will also change. A quantity such as Φ_e/c^2 has been carried forward from barycentric coordinates, and one might question whether it should change due to time and length scale changes. However, such quantities are already of order c^{-2}, and any such changes would be of higher order and are therefore negligible.

Appendix C
Equations of Motion of the Earth and Moon

The equations of motion of the Earth and Moon should be checked to verify that, neglecting solar tidal forces, they orbit around each other in eccentric Keplerian ellipses. The equation of motion of the Earth, using coordinate time x^0 as the independent variable, is

$$\frac{d^2x_e^i}{(dx^0)^2} + \Gamma_{\mu\nu}^i\frac{dx_e^\mu}{dx^0}\frac{dx_e^\nu}{dx^0} - \Gamma_{\mu\nu}^0\frac{dx_e^\mu}{dx^0}\frac{dx_e^\nu}{dx^0}\frac{dx_e^i}{dx^0} = 0. \quad \text{(C1)}$$

The only Christoffel symbol contribution of order c^{-2} is

$$\Gamma^i_{00} = -\frac{1}{2}\frac{\partial g_{00}}{\partial x^i} = \frac{1}{c^2}\frac{\partial(\Phi_e + \Phi_m)}{\partial x^i}. \quad \text{(C2)}$$

This partial derivative must be evaluated at the Earth's center, which would introduce a singularity. However, a body cannot cause the acceleration of its own center of mass, so the term involving the Earth's potential must be "effaced," or discarded. The equation of motion of the Earth then becomes

$$\frac{d^2x_e^i}{(dx^0)^2} - \frac{GM_m(x_e^i - x_m^i)}{c^2|\boldsymbol{r}_e - \boldsymbol{r}_m|^2} = 0. \quad \text{(C3)}$$

A similar argument for the equation of the Moon gives

$$\frac{d^2x_m^i}{(dx^0)^2} - \frac{GM_e(x_m^i - x_e^i)}{c^2|\boldsymbol{r}_e - \boldsymbol{r}_m|^2} = 0. \quad \text{(C4)}$$

The center of mass of the Earth–Moon system should be at

$$x_{cm}^i = \frac{M_e x_e^i + M_m x_m^i}{M_T}. \quad \text{(C5)}$$

Taking the corresponding linear combinates of the above equations of motion gives

$$\frac{d^2x_{cm}^i}{(dx^0)^2} = 0, \quad \text{(C6)}$$

thus verifying that the center of mass of the Earth–Moon system is not accelerated in this coordinate system.

Let the vector from the center of the Earth to the center of the Moon be denoted by $\boldsymbol{D}$. Then, taking the difference between the above two equations of motion gives

$$\frac{d^2\boldsymbol{D}}{(dx^0)^2} - \frac{GM_T\boldsymbol{D}}{c^2D^3} = 0, \quad \text{(C7)}$$

where the distance between Earth and the Moon is given by Equation (19), and $n^2a^2 = GM_T/a$. Then

$$\dot{D} = \frac{nae\sin(f)}{\sqrt{1-e^2}} \quad \text{and} \quad \ddot{D} = \frac{GM_T e\cos(f)(1+e\cos(f))^2}{a^2(1-e^2)^2}. \quad \text{(C8)}$$

The Earth–Moon system satisfies Kepler's equation in the plane of the Earth–Moon orbit:

$$\boldsymbol{D} = D[\cos(f), \sin(f)],$$
$$\times\dot{\boldsymbol{D}} = \frac{na}{\sqrt{1-e^2}}[-\sin(f), \cos(f) + e], \quad \text{(C9)}$$

$$\ddot{D} = \frac{GM_T}{D^2}[-\cos(f), -\sin(f)] = \frac{GM_T\boldsymbol{D}}{D^3}. \quad \text{(C10)}$$

In summary, we have constructed a locally inertial, freely falling frame of reference with the origin at the center of mass of the Earth and Moon and have shown that the Earth and Moon revolve about their mutual center of mass in a Keplerian orbit. The coordinates are not normal Fermi coordinates in the sense that the Christoffel symbols of the second kind are not zero at the origin of the coordinates when calculated in these coordinates. This is because the geodesic along which the origin falls does not account for forces on a test particle at the origin due to Earth and the Moon—only forces due to the Sun are accounted for.

Appendix D
Comparing Results in Rotating and Nonrotating Coordinate Systems

We calculate the fractional difference between a clock on the Moon's surface and a clock on the Earth's surface in three different coordinate systems. These are (1) the center-of-mass locally inertial system, (2) a rotating system in which the x-axis is along the Earth–Moon line, and (3) a translated, rotating system in which the Earth is at the origin of the coordinates and the Earth–Moon line is in the x'-direction. We show that in all three coordinate systems, the fractional rate difference is the same. The Earth–Moon system is assumed to have a Keplerian orbit. To simplify the calculations, we assume that the clocks are on the surfaces of the respective bodies and along the line joining the centers of the Earth and the Moon. This approximation can be refined when the actual positions of the clocks are specified.

D.1. Center-of-mass Inertial Coordinate System

The scalar invariant in the locally inertial frame whose origin is at the center of mass of the Earth–Moon system, neglecting tidal terms, is

$$-ds^2 = -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2\Phi_m}{c^2}\right)(cdT)^2 + \left(1 - \frac{2\Phi_e}{c^2} - \frac{2\Phi_m}{c^2}\right)(dX^2 + dY^2 + dZ^2). \quad \text{(D1)}$$

We use capital letters to denote coordinates in the center-of-mass system. Anticipating that all velocities are small compared to the speed of light and that the calculations are carried out only to order $1/c^2$, the scalar invariant can be written

$$ds = \left(1 + \frac{\Phi_e}{c^2} + \frac{\Phi_m}{c^2} - \frac{(V_x^2 + V_y^2 + V_z^2)}{2c^2}\right)cdT. \quad \text{(D2)}$$

For a clock on the Moon,

$$V_x = \frac{d}{dt}\left(\frac{DM_e}{M_T}\right), \quad V_y = \frac{df}{dt}\frac{DM_e}{M_T}. \quad \text{(D3)}$$

Then, using Equation (20),

$$V_x^2 + V_y^2 = \frac{M_e^2 GM_T(1 + 2e\cos(f) + e^2)}{2c^2M_T^2 a(1-e^2)}. \quad \text{(D4)}$$

Then the proper time on a clock on the Moon during a coordinate time interval dT is

$$d\tau_m = \left(1 - \frac{GM_e}{c^2D} - \frac{GM_m}{c^2R_m} - \frac{M_e^2 GM_T(1 + 2e\cos(f) + e^2)}{2c^2M_T^2 a(1-e^2)}\right)dT. \quad \text{(D5)}$$

For a clock on Earth,

$$V_x = \frac{d}{dt}\left(-\frac{DM_m}{M_T}\right), \quad V_y = -\frac{df}{dt}\left(\frac{DM_m}{M_T}\right). \quad \text{(D6)}$$

Then

$$V_x^2 + V_y^2 = \frac{M_m^2 GM_T(1 + 2e\cos(f) + e^2)}{M_T^2 a(1 - e^2)}, \tag{D7}$$

and the proper time elapsed during a coordinate time interval dT is

$$d\tau_e = \left(1 - \frac{GM_e}{c^2 R_e} - \frac{GM_m}{c^2 D} - \frac{M_m^2 GM_T(1 + 2e\cos(f) + e^2)}{2c^2 M_T^2 a(1 - e^2)}\right)dT. \tag{D8}$$

The fractional difference is

$$\frac{d\tau_m - d\tau_e}{d\tau_e} = -\frac{GM_e}{c^2 D} - \frac{GM_m}{c^2 R_m} + \frac{GM_e}{c^2 R_e} - \frac{(M_e^2 - M_m^2)(1 + 2e\cos(f) + e^2)}{2c^2 M_T^2 a(1 - e^2)}. \tag{D9}$$

The difference in the last term represents a difference of squares of velocities.

D.2. Rotating Center-of-mass Coordinates

Introduce a rotating system with an Earth–Moon line along the new x-axis:

$$X = x\cos(f) - y\sin(f),\quad Y = x\sin(f) + y\cos(f),\quad T = t. \tag{D10}$$

Then

$$dX^2 + dY^2 = dx^2 + dy^2 + \frac{df^2}{dt}(x^2 + y^2)dt^2 + 2\frac{df}{dt}dt(xdy - ydx). \tag{D11}$$

The scalar invariant becomes

$$-ds^2 = -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2\Phi_m}{c^2} - \left(\frac{df}{dt}\right)^2(x^2 + y^2)\right)(cdt)^2 + 2\frac{df}{dt}dt(xdy - ydx) + dx^2 + dy^2 + dz^2. \tag{D12}$$

For a clock on the Moon,

$$x_m = \frac{M_e D}{M_T},\quad y_m = 0;\quad \dot{x}_m = \frac{M_e \dot{D}}{M_T},\quad \dot{y}_m = 0. \tag{D13}$$

For a clock on the Moon, there is no contribution from the Sagnac term. The proper time interval is

$$d\tau_m = \left(1 - \frac{GM_e}{c^2 D} - \frac{GM_m}{c^2 R_m} - \left(\frac{df}{dt}\right)^2 \frac{M_e^2 D^2}{2c^2 M_T^2} - \frac{M_e^2 \dot{D}^2}{2c^2 M_T^2}\right)dt = \left(1 - \frac{GM_e}{c^2 D} - \frac{GM_m}{c^2 R_m} - \frac{M_e^2 GM_T(1 + 2e\cos(f) + e^2)}{2c^2 a(1 - e^2)M_T^2}\right)dt. \tag{D14}$$

Note that there is a significant contribution from the centrifugal potential. For a clock on Earth,

$$x = -D\frac{M_m}{M_T};\quad \dot{x} = -\dot{D}\frac{M_m}{M_T};\qquad y = 0;\quad \dot{y} = 0. \tag{D15}$$

The proper time interval is then

$$d\tau_e = \left(1 - \frac{GM_e}{c^2 R_e} - \frac{GM_m}{c^2 D} - \left(\frac{df}{dt}\right)^2 \frac{1}{2c^2} \times\left(-D\frac{M_m}{M_T}\right)^2 - \frac{1}{2c^2}\left(-\dot{D}\frac{M_m}{M_T}\right)^2\right)dt. \tag{D16}$$

The fractional proper time interval difference reduces exactly to the expression in Equation (D9).

D.3. Rotating Coordinates with Earth at the Origin

For this system, the velocity of the Moon is the relative velocity. This implies the use of a coordinate system in which the Earth is not moving. This has to be a rotating coordinate system with its origin coinciding with the Earth's center. Therefore, translating the origin to the center of the Earth, with no change in the time variable,

$$x = x' - D\frac{M_m}{M_T};\quad dx = dx' - \dot{D}dt\frac{M_m}{M_T};\quad y = y';\quad dy = dy';\quad z = z'. \tag{D17}$$

The scalar invariant becomes

$$-ds^2 = -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2\Phi_m}{c^2} - \left(\frac{df}{cdt}\right)^2 \times\left(\left(x' - D\frac{M_m}{M_T}\right)^2 + y'^2\right)\right)(cdt)^2 + 2\frac{df}{dt}dt\left(\left(x' - D\frac{M_m}{M_T}\right)dy' - y'\left(dx' - \dot{D}\frac{M_m}{M_T}dt\right)\right) + (1 + ...)\left(\left(dx' - \dot{D}\frac{M_m}{M_T}dt\right)^2 + dy'^2 + dz'^2\right). \tag{D18}$$

The potentials in the last term have been suppressed since they do not contribute to the order of this calculation. For a clock on the surface of the Moon,

$$x' = D;\quad \dot{x}' = \dot{D}\quad \text{(radial velocity)}\quad y' = 0;\quad \dot{y}' = 0. \tag{D19}$$

There is no contribution from the Sagnac term, but there is a significant contribution from the centrifugal potential, representing the transverse velocity of the Moon. The radial velocity of the Moon comes from the spatial part of the metric. The proper time interval for such a clock is

$$d\tau_m = \left(1 - \frac{GM_e}{c^2 D} - \frac{GM_m}{c^2 R_m} - \frac{1}{2c^2}\left(\frac{df}{dt}\right)^2\left(D - D\frac{M_m}{M_T}\right)^2\right) \times dt + \frac{1}{2c^2}\left(\dot{D} - \dot{D}\frac{M_m}{M_T}\right)^2 dt = \left(1 - \frac{GM_e}{c^2 D} - \frac{GM_m}{c^2 R_m} - \frac{M_e^2 G M_T (1 + 2e\cos(f) + e^2)}{2c^2 a(1 - e^2) M_T^2}\right) dt. \tag{D20}$$

For a clock on the surface of the Earth,

$$x' = y' = \dot{x}' = \dot{y}' = 0. \tag{D21}$$

The proper time interval is

$$d\tau_e = \left(1 - \frac{GM_e}{c^2 R_e} - \frac{GM_m}{c^2 D} - \left(\frac{df}{dt}\right)^2 \frac{1}{2c^2}\left(-D\frac{M_m}{M_T}\right)^2 - \frac{1}{2c^2}\left(-\dot{D}\frac{M_m}{M_T}\right)^2\right) dt = \left(1 - \frac{GM_e}{c^2 R_e} - \frac{GM_m}{c^2 D} - \frac{M_m^2 G M_T (1 + 2e\cos(f) + e^2)}{2c^2 M_T^2 a(1 - e^2)}\right) dt. \tag{D22}$$

It is easily seen that the fractional proper time difference reduces to previously derived expressions. Thus, the fractional proper time difference is the same in all three coordinate systems.

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